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ON KNASTER'S CONJECTURE

BY

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ABSTRACT. Knaster's conjecture is: given a continuous $g: S^n \to E^m$ and a set Δ of $n-m+2$ distinct points $(q_1, \cdots, q_{n-m+2})$ in S^n there exists a rotation $r: S^n \to S^n$ such that

$$g(r(q_1)) = g(r(q_2)) = \cdots = g(r(q_{n-m+2})).$$

We prove a stronger statement about a smaller class of functions. If $f: S^n \to E^n$ we write $f = (f_1, f_2, \cdots, f_n)$ where $f_i: S^n \to E^1$, and put $F_i = (f_1, \cdots, f_i): S^n \to E^i$ so that $F_n = f$. The level surface of F_i in S^n containing x is $l_i(x) = \{y \in S^n | F_i(x) = F_i(y)\}$.

Theorem. *Given an $(n+1)$-frame $\Delta \subset S^n$ and a real-analytic function $f: S^n \to E^n$ such that each $l_i(x)$ is either a point or a topological $(n-i)$-sphere, there exist at least 2^{n-1} distinct rotations $r: S^n \to S^n$ such that*

$$f_i(r(q_1)) = \cdots = f_i(r(q_{n-i+2})), \quad i = 1, 2, \cdots, n,$$

for each rotation. It follows that for $m = 1, 2, \cdots, n$,

$$F_m(r(q_1)) = F_m(r(q_2)) = \cdots = F_m(r(q_{n-m+2})),$$

so that the functions $F_m: S^n \to E^m$ satisfy Knaster's conjecture simultaneously.

Given F_i, the definition of f can be completed in many ways by choosing $f_{i+1}, \cdots, f_n$, each way giving rise to different rotations satisfying the Theorem. A suitable homotopy of f which changes f_n slightly will give locally a continuum of rotations r each of which satisfies Knaster's conjecture for F_{n-1}. In general there exists an $(n-m)$-dimensional family of rotations satisfying Knaster's conjecture for F_m.

1. **Introduction.** In 1947 Knaster [12] made the following conjucture on mappings of spheres into euclidean space: given a continuous function $g: S^n \to E^m$ and an arbitrary set or frame $(q_1, q_2, \cdots, q_{n-m+2})$ of $n-m+2$ distinct points in S^n, there exists a rotation $r: S^n \to S^n$ such that

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$$g(r(q_1)) = g(r(q_2)) = \cdots = g(r(q_{n-m+2})).$$

This was a broad generalization of the Borsuk-Ulam Theorem ($n = m$, two antipodal points) [1], and Kakutani's Theorem ($n = 2, m = 1$, three orthogonal points) [11].

Subsequently, various special cases have been verified. The principal results have been given by Yamabe and Yujobô (all n, $m = 1$, $n + 1$ orthogonal points) [14], Bourgin (special values of m and n, $n - m + 2$ equispaced points) [2], and Floyd ($n = 2$, $m = 1$, three arbitrary points) [7]. Cases for which there exist families of suitable rotations have been found by Conner and Floyd [5], Bourgin [3] and Cairns [4].

In this paper we prove a statement stronger than that of the original conjecture, but about a smaller class of functions. We denote by S^n the standard n-sphere, by $\Delta = (q_1, \cdots, q_{n+1})$ a fixed frame of $n + 1$ distinct points in S^n, and put $\Delta_i = (q_1, \cdots, q_i)$. If $f: S^n \to E^n$ we write $f = (f_1, \cdots, f_n)$ where $f_i: S^n \to E^1$ and put $F_i = (f_1, \cdots, f_i): S^n \to E^i$ so that $F_n = f$. If

$$l_i(x) = \{y \in S^n \mid F_i(x) = F_i(y)\},$$

then $l_i(x)$ is the level surface containing x of F_i in S^n; it is ordinarily of dimension $n - i$. The purpose of this paper is to prove the following theorem.

Theorem 1. *Given $\Delta \subset S^n$ and a real-analytic function $f: S^n \to E^n$ such that for each $x \in S^n$ and each $i = 1, 2, \cdots, n$, $l_i(x)$ is either a topological $(n - i)$-sphere or a single point, there exist at least 2^{n-1} distinct frames of the form $(p_1, \cdots, p_{n+1})$, each the image of Δ under a rotation of S^n, such that*

$$f_i(p_1) = f_i(p_2) = \cdots = f_i(p_{n-i+2}), \qquad i = 1, 2, \cdots, n.$$

If $\operatorname{diam}(\Delta) < \operatorname{diam}(S^n)$ *there exist at least* 2^n *such frames.*

Another way of expressing this theorem is to consider the functions $F_i: S^n \to E^i$ and the subframes Δ_{n-i+2} of Δ. Then for $i = 1, 2, \cdots, n$ the subframe $(p_1, \cdots, p_{n-i+2})$ of any frame $(p_1, \cdots, p_{n+1})$ whose existence is given by the theorem satisfies

$$F_i(p_1) = F_i(p_2) = \cdots = F_i(p_{n-i+2}).$$

This is the conclusion of Knaster's conjecture for $m = n$, $n = i$, $n - i + 2$ arbitrary points, and with the analytic function F_i restricted so that its level surfaces are topological spheres. The result shows that when the functions are this simple a frame $(p_1, \cdots, p_{n+1})$ exists which satisfies Knaster's conjecture not only for $f = F_n$, but for all of the functions F_i simultaneously.

Theorem 1 is in one sense the best possible result. If E^{n+1} has coordinates

$x_1, x_2, \cdots, x_{n+1}$, and $f_i = x_i$ on $S^n \subset E^{n+1}$ then there exist exactly 2^n frames satisfying the conclusion if diam $(\Delta) <$ diam (S^n) and exactly 2^{n-1} such frames if diam $(\Delta) =$ diam (S^n).

The method of proof is to consider i-frames "similar" to Δ_i which satisfy the conclusions of the theorem. We shall find continuous families of such frames which contain both minimal and maximal members, and which must then contain members of the given size of Δ. These families of "similar" frames are described by certain multiple-valued functions, or relations. In §2 we describe these complete relations, which are used extensively in the proof.

2. **Complete relations.** Let X, Y be Hausdorff spaces and $R: X \to Y$ be a relation with domain X and range Y; that is, $R \subset X \times Y$ and for each $x \in X$ there exists at least one $y \in Y$ such that $(x, y) \in R$. We write as for functions

$$R(x) = \{y \in Y \mid (x, y) \in R\},$$

and if $A \subset X$

$$R(A) = \{y \in Y \mid \text{for some } x \in A, (x, y) \in R\}.$$

We will assume that $R(x)$ is a finite set for all $x \in X$.

A point $(x, y) \in R$ will be called an *ordinary point* if there exist neighborhoods $U(x) \subset X$, $V(y) \subset Y$ such that $R \mid U: U \to V$, where both the domain and range of R have been restricted, is a continuous function. Any other point of R will be called a *tangent point*. We denote the set of all ordinary points of R by R_0, tangent points by R_t, so that $R = R_0 \cup R_t$. It is clear that R_0 is open in R. If $(x, y) \in R_t$ and $(x, y) \in \partial R_0$, then (x, y) will be called a *boundary tangent point*. We denote the set of boundary tangent points by B and note that $B \subset R_t$.

Definition. A *complete relation* $R: X \to Y$ is a finite valued relation such that

(i) R is compact in $X \times Y$,

(ii) for each $x \in X$ and neighborhood $U'(x) \subset X$ there exists a neighborhood $U(x) \subset U'(x)$ such that $R_0 \cap (U \times Y)$ has only a finite number of components,

(iii) denoting a product neighborhood of (x, y) by $U \times V$, projection into X by the subscript X, and $U - (B \cap (U \times V))_X$ by U_B we require: given $(x, y) \in B$ and $U' \times V'$ any neighborhood of (x, y) there exists a neighborhood $U \times V \subset U' \times V'$ of (x, y) containing no other tangent point with first coordinate x and such that either (a) for each $x' \in U_B$, $R_0(x') \cap V$ has even cardinality, or (b) for each $x' \in U_B$, $R_0(x') \cap V$ has odd cardinality.

An example of a complete relation is obtained by considering a 2-manifold without boundary smoothly immersed in E^3 in such a way that the inverse of the projection map into the xy-plane is finite-valued. Then if X is the image of this projection the immersed 2-manifold forms the graph of a complete relation R from

X into E^1. The points of R whose tangent plane is parallel to the z-axis, together with the self-intersection points of the immersion, are the tangent points of R. Also, any continuous function is a complete relation with no tangent points.

We next assign a multiplicity to each point x in the domain of R. This is done by counting in the correct way the number of points of R whose first coordinate is X.

Definition. The *multiplicity* $m(x)$ of a point $x \in X$ with respect to a complete relation R is the sum modulo 2 of the multiplicities of all elements $(x, y) \in R$ where these are defined as follows:

(a) if $(x, y) \in R_0$ its multiplicity is one,

(b) if $(x, y) \in R_t - B$ its multiplicity is zero,

(c) if $(x, y) \in B$ there exists by condition (iii) of the definition a neighborhood $U \times V$ of (x, y) such that if $x' \in U_B$ then the cardinality of $R_0(x') \cap V$ is always either even or odd; if it is odd we put $m(x, y) = 1$, if even $m(x, y) = 0$.

Lemma 1. *If* $R: X \to Y$ *is a complete relation,* X *is connected, and* $x_1, x_2 \in X$, *then* $m(x_1) = m(x_2)$.

Proof. If B_X is the projection of B into X, then for all x in any component Q of $X - B_X$ the multiplicity is the same. It is equal to the number mod 2 of components of R_0 whose projection covers Q. Therefore, by compactness of R it is sufficient to examine the multiplicity of points in a neighborhood of a fixed point $x \in B_X$. There is no loss of generality in assuming that $R(x)$ contains exactly one point $(x, y) \in B$, for if there is more than one point they can be put in disjoint neighborhoods in $X \times Y$ and treated separately.

By condition (iii) of the definition there exists a neighborhood $U \times V$ of (x, y) such that if $x' \in U_B = U - (B \cap (U \times V))_X$, then $m(x')$ is always either even or odd. Suppose it is always even. Then $m(x, y) = 0$ and $m(x) = 0$. Thus if $x' \in U_B$ we have $m(x') = m(x)$ and m is, therefore, a continuous function. Hence, m is constant on X and the proof is complete.

We now define the *multiplicity of a complete relation* $R: X \to Y$ to be either even or odd depending upon whether $m(x)$, for any $x \in X$, is zero or one. Lemma 1 shows that the multiplicity of a complete relation is well defined when X is connected.

Lemma 2. *If* $R: X \to Y$ *is a complete relation, then*

(a) *each component of* R *is a complete relation,*

(b) *if* $A \subset X$ *is compact, then* $R \mid A$ *is a complete relation with the same multiplicity as* R,

(c) *if* R *has odd multiplicity, then it has a component which is a complete relation of odd multiplicity with domain* X.

Proof. Parts (a) and (b) follow immediately from the definition of complete relation. For part (c) we note that the multiplicity is additive mod 2, that is, if R is written as the union of its components $R = R_1 \cup \cdots \cup R_k$, then $m(x) = m_1(x) + \cdots + m_k(x)$ mod 2. Then each component with odd multiplicity must have domain X, and there must be exactly an odd number of such components.

3. **Continuous families of frames.** Recall that we are given S^n, Δ, and $f: S^n \to E^n$. Further $f = (f_1, f_2, \cdots, f_n)$, and we defined $F_i = (f_1, \cdots, f_i)$ and $l_i(x) = \{y \mid F_i(x) = F_i(y)\}$. We next define certain decomposition spaces of S^n denoted by $L^n, L^{n-1}, \cdots, L^1$ as follows.

First define the space

$$L^{n'} = \{l_n \mid l_n \subset S^n\},$$

with topology induced by the function $F_n: S^n \to E^n$ (since F_n is constant on each element of $L^{n'}$ it can be regarded as a function $F_n: L^{n'} \to E^n$). Consider all elements of $L^{n'}$ which consist of a pair of antipodal points. Take a fixed small open neighborhood N of this subset and put $L^n = L^{n'} - N$. Due to the restrictions on F_n, this set is nonempty. Then define, for $i = 1, 2, \cdots, n-1$,

$$L^i = \{l_i \subset S_n \mid l_i \cap (\bigcup \{l_n \mid l_n \in N\}) = \emptyset\}$$

with topology induced by $F_i: S^n \to E^i$. An arbitrary element of L^k will be denoted by l_k; by assumption any such element is either a point or an $(n-k)$-sphere.

We note that L^1, the space of $(n-1)$-spheres that are level surfaces of F_1, is mapped homeomorphically by $F_1: L^1 \to E^1$ to the union of disjoint closed intervals in E^1. Similarly, L^k is homeomorphic to a k-cell with holes. Finally, suppose for the moment that l_k is a fixed nondegenerate element of L^k and consider the space $A = \{l_{k+1} \mid l_{k+1} \subset l_k\} \subset L^{k+1}$. We find that A is homeomorphic to a closed interval by $f_{k+1}: A \to E^1$, since f_{k+1} is constant on each element of A. The exclusion of N is done to eliminate an awkward special case. We will later (§4) remove this restriction and allow N to be empty.

We will assume that the points of Δ are linearly independent, that is, that Δ is not contained in any great $S^{n-1} \subset S^n$. This restriction will be removed at the end of the proof. Geodesic distance in S^n will be denoted by d and the angle between two geodesics xy and xz at x by $\theta(xy, xz)$. The points of Δ are labeled so that

$$d(q_1, q_2) \geq d(q_1, q_3) \geq \cdots \geq d(q_1, q_{n+1}).$$

Definition. The i-frame $(p_1, \cdots, p_i)$ is *similar* to Δ_i ($\sim \Delta_i$) with respect to q_1 if the geodesic distances and angles satisfy

$$d(p_1, p_j)/d(q_1, q_j) = \rho, \qquad j = 2, \cdots, i,$$

$$\theta(p_1 p_j, p_1 p_k) = \theta(q_1 q_j, q_1 q_k), \qquad j, k = 2, 3, \cdots, i.$$

We require for this definition that the point q_1 is not antipodal to any other point of Δ. This follows from the assumed linear independence of the points of Δ. If $i = n + 1$ we also require that any two similar frames induce the same orientation on S^n. Then if Δ is defined by giving the point q_1 and the goedesics $q_1 q_i$, a similar frame is obtained by shrinking the frame along all of the geodesics by a factor ρ with q_1 as a fixed point, and following this by any rotation of S^n.

The plan of the proof is to construct for $i = 2, 3, \cdots, n+1$ a distinguished set of i-frames $\{(p_1, \cdots, p_i)\} \subset S^n \times \cdots \times S^n$ such that

(1) each i-frame is similar to Δ_i,

(2) for each frame, $p_1, p_2, \cdots, p_i \in l_{n-i+2}(p_1)$,

(3) each i-frame, with p_i deleted is a member of the distinguished set of $(i-1)$-frames.

To this end we shall construct complete relations R_k: $L^{n-k+2} \rightarrow S^{n,k}$ ($k = 2, 3, \cdots, n+1$) where $S^{n,k}$ is an identification space of the product $S^n \times \cdots \times S^n$ (k factors). The image of an element $l_{n-k+2} \in L^{n-k+2}$ is a finite set of equivalence classes of k-frames. Each of these frames is a subset of l_{n-k+2} (property (2) above) and also satisfies properties (1) and (3). The situation is described in the following lemma, where we denote an arbitrary element of $R_i(l_{n-i+2})$ by ϕ_i.

Lemma 3. *For $i = 2, 3, \cdots, n+1$ there exists a complete relation of odd multiplicity R_i: $L^{n-i+2} \rightarrow S^{n,i}$ such that*

(a) *for each $l_{n-i+2} \in L^{n-i+2}$, each $\phi_i \in R_i(l_{n-i+2})$ is an equivalence class of i-frames $(p_1, \cdots, p_i)$ each similar to Δ_i, such that for each i-frame, $p_1, p_2, \cdots, p_i \in l_{n-i+2}$ and if $i > 2$ there exists $l_{n-i+3} \subset l_{n-i+2}$ such that $(p_1, p_2, \cdots, p_{i-1}) \in \phi_{i-1} \in R_{i-1}(l_{n-i+3})$,*

(b) *each $(p_1, \cdots, p_i) \in \phi_i \in R_i(l_{n-i+2})$ induces the same orientation on l_{n-i+2}.*

Proof. The proof is by induction on i, the number of points in the frame.

We first note that the nondegenerate level spheres l_j can all be given consistent natural orientations. To define them we first choose a fixed orientation for S^n, and then select a point x where the gradient vectors in S^n of $f_1, f_2, \cdots, f_{n-1}$ exist and are linearly independent. We orient the 1-sphere $l_{n-1}(x)$ by choosing a vector v in l_{n-1} at x which together with these $n-1$ gradient vectors induces an orientation on S^n agreeing with the fixed orientation. All other members of the family $\{l_{n-1}\}$ are oriented by continuity from this one. To orient $l_j(x)$ we use the vector v together with the gradients of $f_{n-1}, f_{n-2}, \cdots, f_{j+1}$ at x, which provide an ordered set of $n - j$ linearly independent vectors in $l_n(x)$ at x and therefore orient this $(n - j)$-sphere. The other members of $\{l_j\}$ are oriented by continuity. In the case where $l_n \in L^n$ is a zero-sphere, the natural orientation will be the ordered pair such that the gradient vectors to $f_1, f_2, \cdots, f_n$ disagree with the given orientation of S^n at the first point and agree at the second point.

We will use this orientation to distinguish between different families of frames. If $(p_1, \cdots, p_i)$ is an i-frame in l_{n-i+2}, then $p_1, p_2, \cdots, p_i$ can be used as the vertices of a triangulation of the $(i - 2)$-sphere l_{n-i+2} which induces an orientation on it. This orientation either agrees or disagrees with the natural orientation defined above, so the nondegenerate i-frames on l_{n-i+2} are divided into two classes. A *degenerate* frame has the form $(p_1, p_1, \cdots, p_1)$; we will assume that any condition placed on the orientation of a degenerate frame in a degenerate sphere is automatically satisfied.

A. **The case** $i = 2$. In this case R_2 is a function and $S^{n2} = S^n \times S^n$. We define

$$R_2: L^n \to S^{n2},$$

$$R_2(l_n) = (p_1, p_2), \text{ or } (p_1, p_1)$$

where $l_n = \{p_1, p_2\}$ and the ordered pair (p_1, p_2) agrees with the natural orientation of l_n. If l_n consists of a single point, then $R_2(l_n)$ is the degenerate frame (p_1, p_1). Note that there are two possibilities for the choice of R_2, depending on whether the image pairs are chosen to agree or disagree with the natural orientation of l_n. It is clear that R_2 is a continuous function, and is therefore a complete relation of odd multiplicity.

B. **The case** $i = 3$. Consider a fixed 1-sphere $l_{n-1} \in L^{n-1}$. R_2 provides a continuous family of 2-frames which sweeps out l_{n-2} with degenerate frames occuring at the two extreme points of f_n in l_{n-1}. If $l_n \subset l_{n-1}$ and $R_2(l_n) = (p_1, p_2)$ we define

$$S^{n-2}(p_1, p_2) = \{p_3 \in S^n \mid (p_1, p_2, p_3) \sim \Delta_3)\}$$

and

$$P_3(l_{n-1}) = \bigcup \{S^{n-2}(p_1, p_2) \mid (p_1, p_2) \in R_2(l_{n-1})\}.$$

The first set is an $(n - 2)$-sphere (degenerate if $p_1 = p_2$) since by similarity p_3 is a fixed distance from p_1 and the geodesic p_1p_3 is at a fixed angle from p_1p_2. The second set is the set of all points p_3 such that for some $(p_1, p_2) \in R_2(l_{n-1})$, (p_1, p_2, p_3) is similar to Δ_3. We shall be interested in $P_3(l_{n-1}) \cap l_{n-1}$, for this is the set of third points of suitable 3-frames inscribed in l_{n-1}.

The elements $l_n \in L^n$ whose union is l_{n-1} can be indexed with the value which f_n takes upon them. Suppose that $l_{n,m}$ and $l_{n,M}$ are two degenerate elements of L^n at which f_n attains respectively its minimum and maximum in l_{n-1}. In a sufficiently small neighborhood of $l_{n,m}$ (or $l_{n,M}$), $P_3(l_{n-1})$ is a slightly distorted 1-sheeted cone with vertex $l_{n,m}$ ($l_{n,M}$). The curve l_{n-1}, in this neighborhood, is essentially the axis of this cone. It follows that for l_n close to $l_{n,m}$ in l_{n-1}, $S^{n-2}(R_2(l_n))$ links l_{n-1} in S^n and $S^{n-2}(R_2(l_n)) \cap l_{n-1} = \emptyset$. $P_3(l_{n-1})$ is the image of an $(n-1)$-sphere mapped into S^n. The complete relation R_3 will stem from sets of the form $P_3(l_{n-1}) \cap l_{n-1}$. We first define a complete relation R_3' of which R_3 will be a subset.

B1. **Definition of R_3'.** We will use the notation $\delta_j = (p_1, p_2, \cdots, p_j)$ for a j-frame. The 3-frames of interest are selected by the relation

$$T_3: L^{n-1} \to S^n \times S^n \times S^n,$$

$$T_3(l_{n-1}) = \{\delta_3 | \text{for some } l_n \subset l_{n-1}, \delta_2 \in R_2(l_n), \text{and } p_3 \in l_{n-1} \cap P_3(l_{n-1})\},$$

where we include the degenerate 3-frame in $T_3(l_{n-1})$ if and only if l_{n-1} is a point. We collect these into equivalence classes by means of the equivalence relation E_3 in $S^n \times S^n \times S^n$ which is defined as follows: $(p_1, p_2, p_3)\, E_3 (p_1', p_2', p_3')$ if either

(a) $p_j = p_j'$ $(j = 1, 2, 3)$, or

(b) (p_1, p_2, p_3) and (p_1', p_2', p_3') lie in the same component of $T_3(l_{n-1})$ for some $l_{n-1} \in L^{n-1}$. Then we put $S^{n,3} = (S^n \times S^n \times S^n)/E_3$. An element of $S^{n,3}$ will be denoted by ϕ_3; it is an equivalence class of 3-frames. We finally define

$$R_3': L^{n-1} \to S^{n,3},$$

$$R_3'(l_{n-1}) = \{\phi_3 | \phi_3 \text{ is a component of } T_3(l_{n-1})\}.$$

By the construction of R_3' it is clear that for each $l_{n-1} \in L^{n-1}$, each $\phi_3 \in R_3'(l_{n-1})$ is an equivalence class of 3-frames such that for each 3-frame, $p_1, p_2, p_3 \in l_{n-1}$ and there exists $l_n \subset l_{n-1}$ such that $(p_1, p_2) \in R_2(l_n)$. To complete the definition of R_3' it remains to say when an element of R_3' is an ordinary element.

In defining the ordinary elements the idea is to consider whether or not, at a point of $R_3'(l_{n-1})$, l_{n-1} pierces $P_3(l_{n-1})$. If so, and if R_3' is locally single-valued, then the point is an ordinary point of R_3'. Otherwise it is a tangent point.

Suppose that $\phi_3 \in R_3'(l_{n-1})$; to specify whether it is an ordinary or tangent point of R_3' we denote by Q the set of all first points of 3-frames in ϕ_3, i.e.,

$$Q = \{p_1 | \exists p_2, p_3 \ni (p_1, p_2, p_3) \in \phi_3\}.$$

Since Q is connected it is a point or a closed interval in l_{n-1}. Let Q' be a closed interval in l_{n-1}, containing Q in its interior and parametrized by t, $[0 \leq t \leq 1]$. We choose Q' so that for $p_1 \in Q' - Q$, $S^{n-2}(l_n(p_1)) \cap l_{n-1} = \emptyset$. We consider the 0-spheres $l_n(t)$ and the associated $(n-2)$-spheres $S^{n-2}(l_n(t))$. The spheres $S^{n-2}(l_n(0))$ and $S^{n-2}(l_n(1))$ either link or do not link l_{n-1}. As t varies from zero to one we consider the linkage change of the spheres $S^{n-2}(l_n(t))$ with l_{n-1} due to the intersection Q. If the linkage changes an odd number of times then (l_{n-1}, ϕ_3) is a *piercing point* of R_3', if an even number of times it is a *non-piercing point*. We can regard a piercing point as corresponding to a piercing of $P_3(l_{n-1})$ by l_{n-1}. We notice that if (l_{n-1}, ϕ_3) is a piercing point, and if l'_{n-1} is sufficiently close to l_{n-1} then l'_{n-1} must pierce $P_3(l'_{n-1})$ exactly an odd number of times near ϕ_3.

We now say that a relation $R: X \to Y$ is *locally single-valued* at $(x, y) \in R$ if there exists a neighborhood $U \times V$ of (x, y) in $X \times Y$ such that $R \mid U: U \to V$, where both domain and range of R are restricted, is a function. We then make the definition: $(l_{n-1}, \phi_3) \in R_3'$ is an *ordinary point* if it is a piercing point and if R_3' is locally single-valued at (l_{n-1}, ϕ_3); otherwise it is a *tangent point* of R_3'.

We must show that the definition of ordinary point is properly made, i.e., if (l_{n-1}, ϕ_3) is an ordinary point then there exists a neighborhood $U \times V$ of (l_{n-1}, ϕ_3) in $L^{n-1} \times S^{n3}$ such that $R_3' \mid U: U \to V$ is a continuous function. We know that there exists a neighborhood $U \times V$ of (l_{n-1}, ϕ_3) such that $R_3' \mid U: U \to V$ is a function. We show that $R_3' \mid U$ is continuous at $l'_{n-1} \in U$. Suppose that $R_3'(l'_{n-1}) = \phi_3'$ and that $V' \subset V$ is a neighborhood of ϕ_3'. Since l_{n-1} pierces $P_3(l_{n-1})$ at ϕ_3 and R_3' is locally single-valued, l'_{n-1} pierces $P_3(l'_{n-1})$ at ϕ_3' and if l''_{n-1} is sufficiently close to l'_{n-1}, then l''_{n-1} pierces $P_3(l''_{n-1})$ at some $\phi_3'' \in V'$. Thus there exists a neighborhood U' of l'_{n-1} such that $R_3'(U') \cap V \subset V'$ and continuity is proved. Hence the ordinary points of R_3' are defined correctly.

B2. **Proof that R_3' is a complete relation.** There are three properties to be verified.

(i) R_3' is compact. It is a closed subset of a compact space.

(ii) For each $l_{n-1} \in L^{n-1}$ and neighborhood $U'(l_{n-1})$ we will find a neighborhood $U(l_{n-1})$ such that $(R_3')_0 \cap (U \times S^{n,3})$ has only a finite number of components. We do this by using the analyticity of the function F. That is, since l_{n-1} and $P_3(l_{n-1})$ can be described by analytic functions, their intersection is the "level surface" of an analytic function on a connected compact set and therefore has a finite number of components.

We first define the relations

$$D_3: L^{n-1} \to R_2(L^n) \times S^n,$$

$$D_3(l_{n-1}) = \{(p_1, p_2, p_3) \mid \text{for some } l_n \subset l_{n-1}, (p_1, p_2) = R_2(l_n), p_3 \in S^{n-2}(p_1, p_2)\},$$

and

$$\Lambda_3: L^{n-1} \to R_2(L^n) \times S^n,$$

$$\Lambda_3(l_{n-1}) = R_2(L^n) \times l_{n-1}.$$

Note that $R_2(L^n)$ is the same space as L^n but with the elements taken as ordered pairs. We see that $D_3(l_{n-1})$ is the set of all 3-frames similar to Δ_3 such that if p_3 is deleted, (p_1, p_2) is a 2-frame in l_{n-1} obtained from R_2. The projection of $D_3(l_{n-1})$ into the third coordinate space S^n is the set $P_3(l_{n-1})$. $\Delta_3(l_{n-1})$ is the set of all 3-frames with $p_3 \in l_{n-1}$ and $(p_1, p_2) \in R_2(L^n)$. We can use those relations to obtain a new expression for T_3,

$$T_3: L^{n-1} \to S^n \times S^n \times S^n,$$

$$T_3(l_{n-1}) = \{(p_1, p_2, p_3) | (p_1, p_2, p_3) \in D_3(l_{n-1}) \cap \Lambda_3(l_{n-1})\},$$

where we include the degenerate frame (p_1, p_1, p_1) in $T_3(l_{n-1})$ if and only if l_{n-1} is a point, namely $l_{n-1} = \{p_1\}$.

These sets can be described by analytic functions in the following way. We define the function

$$g: R_2(L^n) \times S^n \to E^{n-1} \times E^2,$$

$$g(p_1, p_2, p_3) = (F_{n-1}(p_1), \sigma, \phi),$$

where

$$d(p_1, p_3)/d(p_1, p_2) = \sigma, \qquad \theta(p_1p_2, p_1p_3) = \phi.$$

A level surface of this analytic function is a set of 3-frames similar to one another such that if p_3 is deleted, (p_1, p_2) is a 2-frame in l_{n-1} obtained from R_2. If the 3-frames are similar to Δ_3 it is a set $D_3(l_{n-1})$. We also define

$$h: R_2(L^n) \times S^n \to E^{n-1} \times E^2,$$

$$h(p_1, p_2, p_3) = (F_{n-1}(p_3), r, \psi),$$

where $r = d(q_1q_3)/d(q_1q_2)$, $\psi = \theta(q_1q_2, q_1q_3)$. These two numbers are fixed and depend only on Δ. A level surface of this function is a set of the form $\Lambda_3(l_{n-1})$. We now form

$$(g - h): R_2(L^n) \times S^n \to E^{n-1} \times E^2,$$

where the difference means vectorial subtraction in E^{n+1}. The level surface $(g-h)^{-1}(0)$ is the set of all 3-frames inscribed in elements of L^{n-1} which have the desired properties. We can then write

$$T_3(l_{n-1}) = (g-h)^{-1}(0) \cap [R_2(L^n) \times l_{n-1}].$$

Thus the suitable 3-frames inscribed in l_{n-1} are given by the zeros of an analytic function on a set which is itself a level surface of an analytic function. Then $T_3(l_{n-1})$ has only a finite number of components and the finiteness condition on the complete relation R_3' is satisfied.

(iii) Suppose $(l_{n-1}, \phi_3) \in B_3$, the set of boundary tangent points of R_3'. There are three possibilities. (a) (l_{n-1}, ϕ_3) is not a piercing point, (b) (l_{n-1}, ϕ_3) is a piercing point but R_3' is not locally single-valued there, (c) (l_{n-1}, ϕ_3) is degenerate. We consider them separately.

(iii) (a). If l_{n-1} does not pierce $P_3(l_{n-1})$ at ϕ_3 and if l'_{n-1} is sufficiently close to l_{n-1}, then l'_{n-1} must pierce $P_3(l'_{n-1})$ exactly an even number of times (possibly zero) near ϕ_3. We can find a neighborhood $U \times V$ of (l_{n-1}, ϕ_3) such that V contains no other point of B_3 with first coordinate l_{n-1}, and such that $R_3'(U) \cap (U \times V)$ is connected. Then if $l'_{n-1} \in U_B$, $(R_3')_0(l_{n-1}) \cap V$ must have even cardinality. In this case (l_{n-1}, ϕ_3) is a tangent point of multiplicity zero.

(iii) (b). If l_{n-1} pierces $P_3(l_{n-1})$ at ϕ_3 and if l'_{n-1} is sufficiently close to l_{n-1}, then l'_{n-1} must pierce $P_3(l'_{n-1})$ exactly an odd number of times near ϕ_3. Then, choosing a neighborhood $U \times V$ we see that if $(l'_{n-1}) \in U_B$, $(R_3')_0(l'_{n-1}) \cap V$ must have odd cardinality. Here (l_{n-1}, ϕ_3) is a tangent point of multiplicity one.

(iii) (c). If (l_{n-1}, ϕ_3) is degenerate, then $l_{n-1} = \{p_1\}$ and $\phi_3 = (p_1, p_1, p_1)$. By the analyticity of f_n, if l'_{n-1} is nondegenerate and close to l_{n-1} then l_{n-1} is essentially an ellipse. We shall see in the proof of the fact that R_3 has odd multiplicity that if l'_{n-1} is such an ellipse, then $R_3(l'_{n-1})$ contains exactly one ordinary point. Again in this case, if $l'_{n-1} \in U_B$ then $(R_3')_0(l'_{n-1})$ has odd cardinality and (l_{n-1}, ϕ_3) is a tangent point of multiplicity one. This completes the proof that R_3' is a complete relation.

B3. **Definition of** R_3. We must finally obtain a complete subrelation $R_3 \subset R_3'$ of odd multiplicity which satisfies the orientation condition (b). Returning to a fixed element $l_{n-1} \in L^{n-1}$ we consider the family of spheres $B = \{S^{n-2}(l_n) \mid l_n \subset l_{n-1}\}$ whose union is $P_3(l_{n-1})$. These spheres can be indexed with the numbers $f_n(l_n)$ which form a closed interval $[a, b] \subset E^1$. Thus there is a continuous family of spheres indexed by closed interval, with a degenerate sphere at each end.

If $l_{n,m}$ and $l_{n,M}$ are the degenerate elements of L^n in l_{n-1}, then in neighborhoods of these two points in S^n, $P_3(l_{n-1})$ is a cone whose generating spheres link l_{n-1}. We next show that they link l_{n-1} with opposite orientation near the two degenerate points. We fill in each $S^{n-2}(p_1, p_2) \in B$ with a geodesically flat $(n-1)$-disc $D^{n-1}(p_1, p_2)$ in S^n in such a way as to get a continuous family of such discs. There are two possible ways of doing this, and we do it so that near the cone vertices $D^{n-1}(p_1, p_2)$ is small. Then near the cone vertices, l_{n-1} intersects $D^{n-1}(p_1, p_2)$. We take a particular $D^{n-1}(p_1, p_2)$ near $l_{n, M}$ and orient it by choosing $(n-1)$ vectors in D^{n-1} so that these together with a vector in the direction of the oriented l_{n-1} at its point of intersection with D^{n-1} form a set of coordinates that agrees in orientation with the fixed orientation of S^n. The other members of the family of $(n-1)$-discs associated with l_{n-1} are oriented from this one using continuity.

Now we can define a linkage number of any $S^{n-2}(p_1, p_2)$ with l_{n-1} provided that S^{n-2} does not intersect l_{n-1}. Consider the intersections of l_{n-1} with that D^{n-1} whose boundary is S^{n-2}. If the $(n-1)$ orientation vectors in D^{n-1} plus a vector at the point of intersection agree with the orientation of S^n we assign to that intersection the number +1, if they disagree, −1. If l_{n-1} intersects D^{n-1} without piercing, assign the number zero. Then the *linking number* of S^{n-2} with l_{n-1} is the sum of these numbers.

Now for any $\phi_3 \in R_3'(l_{n-1})$ we can assign a *linking* change −1, 0 or +1 depending on whether, as the spheres S^{n-2} sweep through the continuous family from $l_{n, M}$ to $l_{n, m}$, the linking number decreases, remains unchanged or increases at ϕ_3. We see that since the direction of l_{n-1} is reversed with respect to the local D^{n-1} near $l_{n, m}$ from its direction near $l_{n, M}$, that the sum of the linking changes in $R_3'(l_{n-1})$ is always −2. Furthermore, if ϕ_3 is a nonpiercing point of $R_3'(l_{n-1})$ and if l'_{n-1} is close to l_{n-1}, then the sum of the linking changes in $R_3'(l'_{n-1})$ near ϕ_3 must be zero.

If $\phi_3 \in R_3'(l_{n-1})$ and if $(p_1, p_2, p_3) \in \phi_3$, then the triple (p_1, p_2, p_3) induces an orientation on the 1-sphere l_{n-1} which either agrees or disagrees with the given orientation of l_{n-1}. If l_{n-1} is close to a degenerate element, and is therefore nearly an ellipse, $R_3'(l_{n-1})$ will contain exactly two elements, of which one agrees and the other disagrees with the given orientation of l_{n-1}. For the family of spheres B, in order to produce the linking change of −2 in this simple case, must contain exactly two members which intersect l_{n-1}, and they define 3-frames with opposite orientation on l_{n-1}. Any other element of $R_3'(l_{n-1})$ induces an orientation which associates it with one of the two possible classes.

Therefore the nondegenerate frames of $R_3'(l_{n-1})$ are the union of two disjoint sets. We define $R_3 \subset R_3'$ to be the closure in R_3' of the set of elements with positive orientation. Then R_3 is the required complete relation. It has odd multiplicity at the nearly elliptical element l_{n-1}, and hence has odd multiplicity everywhere. Considering the two ways of choosing R_2, there are four possible ways of choosing R_3 so that it satisfies all of the conditions of Lemma 3.

C. The case $i = k$. The proof for this case is largely a repetition of the proof for $i = 3$.

C1. **Definition of** R'. Consider a fixed $(k-2)$-sphere $l_{n-k+2} \in L^{n-k+2}$. We will put $\alpha = n-k+2$, and use $l_{\alpha+1}$ to denote an arbitrary element of $L^{\alpha+1}$ which is contained in the fixed sphere l_α. We will use the notation $\delta_j = (p_1, \cdots, p_j)$. The elements $l_{\alpha+1}$ sweep out l_α, and $\{R_{k-1}(l_{\alpha+1}) \mid l_{\alpha+1} \subset l_\alpha\}$ provides a continuous family of $(k-1)$-frames with a degenerate frame occuring at the extrema of $f_{\alpha+1}$ in l_α. For each $\delta_{k-1} \in \phi_{k-1} \in R_{k-1}(l_{\alpha+1})$ we define the $(n-k+1)$-sphere

$$S^{\alpha-1}(\delta_{k-1}) = \{p_k \mid (\delta_k, p_k) \sim \Delta_k\}.$$

We further define

$$P_k(l_\alpha) = \bigcup \{S^{\alpha-1}(\delta_{k-1}) \mid \text{for some } l_{\alpha+1} \subset l_\alpha,\ \delta_{k-1} \in \phi_{k-1} \in R_{k-1}(l_{\alpha+1})\}.$$

The elements of $P_k(l_\alpha) \cap l_\alpha$ are points which can be used as the kth point of a k-frame in l_α which satisfies the conditions of the lemma.

To construct R_k we define the relation

$$T_k \colon L^\alpha \longrightarrow (S^n)^k,$$

$$T_k(l_\alpha) = \{\delta_k \mid \text{for some } l_{\alpha+1} \subset l_\alpha,\ \delta_{k-1} \in \phi_{k-1} \in R_{k-1}(l_{\alpha+1}),$$
$$\text{and } p_k \in l_\alpha \cap P_k(l_\alpha)\},$$

where we include the degenerate k-frame in $T_k(l_\alpha)$ if and only if l_α is a point. We have not included the restriction that an element of $T_k(l_\alpha)$ must agree in orientation with l_α; this restriction will be added later.

We next define an equivalence relation E_k in $(S^n)^k$ as follows: $(p_1, \cdots, p_k)$ $E_k(p_k', \cdots, p_k')$ if either (a) $p_j = p_j'$ $(j = 1, 2, \cdots, k)$, or (b) $(p_1, \cdots, p_k)$ and $(p_1', \cdots, p_k')$ lie in the same component of $T_k(l_\alpha)$. Then we put $S^{n,k} = (S^n)^k/E_k$. An element $\phi_k \in S^{n,k}$ is thus an equivalence class of k-frames. We finally define

$$R_k' \colon L^\alpha \longrightarrow S^{n,k},$$

$$R_k'(l_\alpha) = \{\phi_k \mid \phi_k \text{ is a component of } T_k(l_\alpha)\}.$$

We will later replace R_k' by a subrelation R_k whose frames induce the natural orientation on l_α. It turns out that R_k' is a complete relation of even multiplicity, and is the union of two complete relations of odd multiplicity which intersect only at degenerate frames. Thus, once R_{k-1} is given, R_k can be chosen in two ways, so that its frames either agree or disagree with the orientation of l_α. Counting the ways in which the relations R_i $(i \leq k)$ can be chosen, there are 2^{k-1} possible ways of defining R_k.

Returning to R_k' and allowing l_α to be an arbitrary element of L^α, it is clear from the construction that for each $l_\alpha \in L^\alpha$, each $\phi_k \in R_k'(l_\alpha)$ is an equivalence class of k-frames such that for each one, $p_1, \cdots, p_k \in l_\alpha$ and there exists $l_{\alpha+1} \subset l_\alpha$ such that $(p_1, \cdots, p_{k-1}) \in \phi_{k-1} \in R_{k-1}(l_{\alpha+1})$.

We next show that R_k' is a complete relation. As in the case $i = 3$, if $\phi_k \in R_k'(l_\alpha)$ we must specify whether it is a tangent or an ordinary point of R_k. We base the definition on whether or not l_α pierces $P_k(l_\alpha)$ at ϕ_k. The situation is somewhat more complicated, for while R_2 is a single-valued complete relation (a function), R_{k-1} is not single-valued. The set

$$M^{\alpha+1} = \{l_{\alpha+1} \mid l_{\alpha+1} \subset l_\alpha\} \subset L^{\alpha+1}$$

can be indexed with the real numbers $f_{\alpha+1}(l_{\alpha+1})$ and is homeomorphic to a closed interval of the reals. Therefore the restriction

$$R_{k-1} \mid M^{\alpha+1}: M^{\alpha+1} \to S^{n,k-1}$$

is a complete relation defined on the homeomorph of a closed interval. Suppose ϕ_k is an element of $R_k'(l_\alpha)$. Each frame $(p_1, \cdots, p_k) \in \phi_k$ corresponds to a sphere $S^{\alpha-1}(p_1, \cdots, p_{k-1})$, which in turn is generated by an element of $R_{k-1} \mid M^{\alpha+1}$. Therefore ϕ_k corresponds to a connected subset Q of $R_{k-1} \mid M^{\alpha+1}$. Let q be the projection of this subset into $M^{\alpha+1}$, and let q' be a connected closed subset of $M^{\alpha+1}$ containing q in its interior; q and q' are both homeomorphic to closed intervals.

We restrict our attention to the component C of $R_{k-1} \mid q'$ which is contained by Q, and choose q' so small that this component does not contain a $(k-1)$-frame corresponding to a point of $R_k'(l_\alpha)$ different from ϕ_k. Suppose that $l'_{\alpha+1}$ and $l''_{\alpha+1}$ are the end elements of the interval q'. We consider $R_{k-1}(l'_{\alpha+1}) \cap C$ and $R_{k-1}(l''_{\alpha+1}) \cap C$. Since R_{k-1} is a complete relation we can regard $C: q' \to S^{n,k-1}$ as a complete relation. Then C has the same multiplicity at both $l'_{\alpha+1}$ and $l''_{\alpha+1}$. Then the set

$$E = \{\phi_{k-1} \mid \phi_{k-1} \in C(l'_{\alpha+1}) \text{ or } \phi_{k-1} \in C(l''_{\alpha+1})\}$$

contains an even number of elements, provided we assume (which we can) that q' has been chosen so that $C(l'_{\alpha+1})$ and $C(l''_{\alpha+1})$ consist only of ordinary points.

We choose one representative $(k-1)$-frame from each $\phi_{k-1} \in E$ and group these representatives arbitrarily into pairs. For each pair we take a path in C from one frame to the other. The points of this path generate a continuous family of $(n-k+1)$-spheres of the form $S^{\alpha-1}(p_1, \cdots, p_{k-1})$. As a moving sphere in S^n sweeps out this family, its linking number with the $(k-2)$-sphere l_α either does not change, or changes (we disregard changes in linkage that occur due to intersections other than those associated with ϕ_k) by ± 1. For each pair of representative $(k-1)$-frames we find the change in linking number. If the algebraic sum of these changes in linking number is odd, we say that $(l_\alpha, \phi_\alpha) \in R'_k$ is a *piercing* point; otherwise it is a *nonpiercing* point.

We remark that the definition is independent of the way the representatives or pairs are chosen. Also, if l'_α is sufficiently close to l_α, and l_α pierces $P_k(l_\alpha)$ at ϕ_k, then l'_α must pierce $P_k(l'_\alpha)$ exactly an odd number of times near ϕ_k.

We now say that an element $(l_\alpha, \phi_k) \in R'_k$ is an *ordinary point* of R'_k if it is a piercing point and if R'_k is locally single-valued; otherwise it is a *tangent point*. The proof that the definition of ordinary point is properly made is the same as in the case $i = 3$.

C2. **Proof that R'_k is a complete relation.** There are three properties to verify

(i) R'_k is compact; it is a closed subset of a compact space.

(ii) the proof is much the same as for the case $i = 3$, using the analyticity of the functions.

We define

$$g: R_{k-1}(L^{\alpha+1}) \times S^n \to (E^\alpha \times E^{k-1}),$$

$$g(\delta_{k-1}, p_k) = (F_\alpha(p_1), \sigma, \phi_1, \phi_2, \cdots, \phi_{k-2}),$$

where

$$\sigma = d(p_1, p_k)/d(p_1, p_2), \qquad \phi_i = \theta(\overline{p_1 p_{i+1}}, \overline{p_1 p_k})$$

and

$$h: R_{k-1}(l^{\alpha+1}) \times S^n \to (E^\alpha \times E^{k-1}),$$

$$h(\delta_{k-1}, p_k) = (F_\alpha(p_k), r, \psi_1, \psi_2, \cdots, \psi_{k-2}),$$

where $r = d(q_1, q_k)/d(q_1, q_2)$, $\psi_i = \theta(\overline{q_1 q_{i+1}}, \overline{q_1 q_k})$.

Then as before

$$T_k(l_\alpha) = (g - h)^{-1}(0) \cap [R_{k-1}(L^{\alpha+1}) \times l_\alpha],$$

and the components of $T_k(l_\alpha)$ are given as level surfaces of an analytic function on a compact set. Then $T_k(l_\alpha)$ has only a finite number of components and the finiteness condition is satisfied.

(iii) Suppose $(l_\alpha, \phi_k) \in B'_k$ the set of boundary tangent points of R'_k. The same three possibilities occur as for $i = 3$. If (l_α, ϕ_k) is a piercing (nonpiercing) point, it is a tangent point of odd (even) multiplicity. If (l_α, ϕ_k) is degenerate it is a tangent point of odd multiplicity.

C3. **Definition of R_k.** We must finally find a subrelation $R_k \subset R'_k$ which is of odd multiplicity and which satisfies the orientation condition (b). We know by induction that if $\delta_k = (p_1, \cdots, p_k) \in \phi_k \in R'_k(l_\alpha)$, then $\delta_{k-1} = (p_1, \cdots, p_{k-1}) \in \phi_{k-1} \in R_{k-1}(l_{\alpha+1})$ for some $l_{\alpha+1} \subset l_\alpha$. Further, δ_{k-1} defines an orientation on the $(k-3)$-sphere $l_{\alpha+1}$ which agrees with its natural orientation, and $l_{\alpha+1}$ separates the $(k-2)$-sphere l_α into two discs.

Now suppose that l_α is close to a degenerate element of L^α, i.e. l_α is nearly an ellipsoid and any $l_{\alpha+1} \subset l_\alpha$ is hence a near-ellipsoid which separates it. Then $R_{k-1}(l_{\alpha+1})$ contains exactly one element for each $l_{\alpha+1} \subset l_\alpha$. The corresponding spheres $S^\alpha(R_{k-1}(l_{\alpha+1}))$ link l_α with opposite orientation near the two degenerate elements of $L^{\alpha+1}$ in l_α. Thus when l_α is sufficiently close to an ellipsoid, $R_k'(l_\alpha)$ contains exactly two elements, say ϕ_k and ϕ_k'. Of these two, one has $F_{\alpha+1}(p_k) > F_\alpha(p_1)$ while the other has $F_{\alpha+1}(p_k) < F_\alpha(p_1)$. Thus ϕ_k and ϕ_k' form the vertices of natural triangulations of l_α which induce opposite orientations of l_α. Only one of these agrees with the given orientation.

If we remove the degenerate frames from R_k', each component of the resulting set consists entirely of frames which induce the same orientation on the $(k-2)$-sphere l_α which contains it. We therefore define R_k to be the closure of the subset of R_k' consisting of points (l_α, ϕ_k) such that the orientation induced on l_α by ϕ_k is the given orientation of l_α. This is a complete subrelation of odd multiplicity, since it has multiplicity one on any near-ellipsoidal element of L^α. We note that there are two possible ways of choosing R_k from R_k'. Thus, considering the 2^{k-2} ways of choosing R_{k-1}, there are 2^{k-1} possible ways of choosing R_k.

This concludes the proof of Lemma 3. We note that by induction there are 2^{n-1} ways of choosing the complete relation R_n. However, there will still be only 2^{n-1} possibilities for R_{n+1}. The reason for this is that one-half of the possible frames involve a reversal of orientation that is ruled out by the original statement of the conjecture which says that only congruences obtained by rotation of S^n are to be considered.

4. **Proof of Theorem 1.** We have defined

$$R_{n+1}: L^1 \to S^{n,n+1}.$$

Recall that L^1 is homeomorphic to the union of a finite number of closed intervals. The open intervals separating them correspond to the neighborhood N which was introduced in the beginning of §3 to define the spaces L^i. We know that R_{n+1} contains degenerate frames at the points on S^n where F_1 attains its extrema. We define a *maximal* $(n+1)$-frame similar to Δ as a similar $(n+1)$-frame such that p_1 and p_2 are antipodal. We will show that as the neighborhood N shrinks to zero radius (and in the limit is empty) at least one maximal frame must appear in R_{n+1}.

To prove this, suppose that for $N = \emptyset$, no maximal frame appears in R_{n+1}. Now each $(n+1)$-frame δ_{n+1} of R_{n+1} defines an n-plane in E^{n+1}, and if no frame is maximal, then no member of the resulting continuous family of n-planes contains the origin. Each n-plane intersects S^n in a true $(n-1)$-sphere $S^{n-1}(\delta_{n+1})$.

Each member of this continuous family of oriented $(n-1)$-spheres induces an orientation on S^n by taking as the interior of $S^{n-1}(\delta_{n+1})$ the smaller of the two components of the complement.

The frames δ_{n+1} of R_{n+1} induce an orientation on S^n in another way. Each $(n+1)$-frame is contained in an $(n-1)$-sphere which is an element of L^1 and agrees with it in orientation. We choose the interior of $l_1 \in L^1$ to be the set $\{x \in S^n \mid F_1(x) < F_1(l_1)\}$, and this induces an orientation on S^n which, say, agrees with the orientation induced by the spheres $S^{n-1}(\delta_{n+1})$.

Now consider two elements l_1' and $l_1'' \in L^1$ which are small $(n-1)$-spheres close to the degenerate spheres of L^1 occurring at the minimum and maximum respectively of F_1 in S^n. The frames $\delta_{n+1}' = R_{n+1}(l_1')$, $\delta_{n+1}'' = R_{n+1}(l_1'')$ yield spheres $S^{n-1}(\delta_{n+1}')$, $S^{n-1}(\delta_{n+1}'')$ which approximate l_1' and l_1'' respectively. Now if $\delta_{n+1} \in R_{n+1}(l_1)$ for some $l_1 \in L^1$ then the two spheres l_1 and $S^{n-1}(\delta_{n+1})$ intersect in the points of δ_{n+1} and can both be triangulated with these points as vertices in exactly the same way. Hence δ_{n+1}' induces the same orientation on both $S^{n-1}(\delta_{n+1}')$ and l_1', and similarly for δ_{n+1}''.

Thus we know that near the minimum of F_1 in S^n, the orientations of l_1 and of $S^{n-1}(\delta_{n+1}')$ agree, that both induce the same orientation on S^n, and that the interior of each sphere is the smaller of the two components of its complement. However, at l_1'' the orientations of l_1'' and $S^{n-1}(\delta_{n+1}'')$ agree, both induce the same orientation on S^n, but the interior of $S^{n-1}(\delta_{n+1}'')$ is the smaller, and that of l_1'' is the larger, of the two components of its complement; this is impossible, and therefore a maximal frame appears in R_{n+1}, as the neighborhood N shrinks to the empty set.

We now know using Lemma 2c that if N is taken sufficiently small, the two degenerate frames of R_{n+1} each belong to a continuous family of frames which contains a maximal element. Therefore, if Δ is not maximal (and its points are linearly independent) then R_{n+1} will contain at least two frames congruent to Δ. Since R_{n+1} can be chosen in any one of 2^{n-1} ways, we see that if the points of Δ are linearly independent there are at least 2^n frames in S^n satisfying the statement of Theorem 1. If Δ is maximal each possible R_{n+1} may produce only one frame congruent to Δ, and the total number of possibilities is then 2^{n-1}.

If the points of Δ are not linearly dependent we choose a sequence $\{\Delta_k\}$ of frames converging to Δ, each of which is linearly independent. Applying Theorem 1 for each of these we obtain for each k a sequence of frames satisfying Theorem 1 and choose a convergent subsequence. The result is a frame satisfying Theorem 1 for the given Δ. In this case, the two possibilities arising for each choice of R_{n+1} may produce sequences which converge to the same frame, so the total number of frames satisfying the theorem is again at least 2^{n-1}.

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